Quantum Mechanics as a Classical Theory XIII: The Tunnel Effect

L.S.F. Olavo
Departamento de Fisica ,Universidade de Brasilia - UnB
70910-900, Brasilia - D.F. - Brazil

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Abstract

In this continuation paper we will address the problem of tunneling. We will show how to settle this phenomenon within our classical interpretation. It will be shown that, rigorously speaking, there is no tunnel effect at all.

1 Introduction

Since the very moment when one is introduced to the quantum formalism *via* Schrödinger equation, he takes contact, as an initiation ritual, with the problems involving those potentials considered simple—those which mathematical solutions is sufficiently feasible, furnishing however some acceptable physical content.

The student is then first exposed to problems related with the "square potentials" (step, barrier, etc.) and only later will face problems like the harmonic oscillator, the hydrogen atom, etc.

It is with the square potentials that he first is put into contact with the intriguing tunneling effects. It is also with these first "naive" applications that it becomes established the fundamental distinction between the theoretical quantum level (where such tunneling effects are expected) and the classical one (where such effects cannot, supposedly, be expected). This first

mental exercise, by its very simplicity, has such a powerful effect over the conceptions of those involved with it that the impressions, left behind by the interpretation usually superimposed on it by the accepted epistemology, remain in the spirit of the initiate for all his future carrier.

This is precisely the motivation for these first steps: the indelible fixation of a determinate world scheme in the spirit of the future generation of scientists—the forge of a consensus. Such exercises, hence, are exemplary and, hardly, "naive".

In the present series of papers [1]-[12] we are trying to show that it is possible to superimpose to the formal apparatus of quantum mechanics a completely classical interpretation, without appealing to anyone of the usual interpretation aspects of the quantum epistemology (e.g. observers, wave-particle duality, etc.).

It is the intention of this paper to elucidate the known problem of the tunnel effect and show that it may also be settled within a purely classical interpretation. It is unnecessary to say that no appeal to the duality aspect of matter or any other scheme not founded on the classical interpretation will be here allowed. We will have to deal only with *ensembles* of classical particles.

To attain this goal, in the second section we will proceed with a quick analysis of the mathematical developments found in the literature. We are interested here with a critical analysis of the square potentials.

In the third section we will present, in a general fashion, the harmonic oscillator problem in the way it is described in the literature. This problem addresses the one of tunneling for it presents, following the usual interpretation, a non-zero probability of finding particles in regions beyond the classical turning points.

In the forth section the same problem will be dealt with following the formalism already developed in a number of previous papers. We will then reveal, with an analysis of the problem on the phase-space, which interpretation is more adequate to this phenomenon. The classical character of this phenomenon will be totally established. It is possible that, after this exposition, those more sensitive, generally inclined to purism, will find it hard to use the denomination "tunneling" to this effect. The vanishing of the nomenclature, however, will reveal the very weakness of the interpretation responsible for its significance.

2 Square Potentials

The first potentials to which we are first presented when initiating the study of quantum theory are, for reasons of mathematical simplicity, those we may classify as square potentials, after their mathematical expression[13, 14, 15]

$$V(x) = \left\{ \begin{array}{l} V0 \text{ if } b \leq x \leq c, \\ 0 \text{ else} \end{array} \right. . \tag{1}$$

The solutions related with the introduction of such potentials are given by

$$\psi(x) = Ae^{ikx} + Be^{-ikx}, \tag{2}$$

being k , eventually, complex (e.g. inside the barrier) and where A and B are constants.

We may at this point present two sources of problems: the first one is related with the expression for the potentials while the second is related with the aspect of the functions which are solutions of the Schrödinger equation resulting from the consideration of these potentials.

Clearly, a potential as the one described in (1) does not pertain to the scope of a quantum formalism as here understood. Indeed, we shall remember that the Schrödinger equation was *derived* from the Liouville equation using the supposition

$$\frac{\partial V(x)}{\partial x}\delta x = V(x + \delta x/2) - V(x - \delta x/2), \tag{3}$$

where δx is an infinitesimal displacement. This supposition requires that the potential has a continuous first order derivative over all its domain of definition.

This constraint is also related with the very use of the potential inside the Liouville equation, where it appears inside the derivative sign.

Since these potentials do not satisfy the *conditio sine qua non* of the quantum formalism, they cannot be considered as adequate to represent situations manageable within the theory. Hence, far from being exemplary for a first study of the formalism, they are, indeed, by no means, representative of the kind of phenomena the theory is apt to reveal.

At this stage, some attentive reader might ask how it is possible to say such a thing if we see at every moment the experiments confirming the behavior raised by the form of these potentials? If, even unexpected from realistic

situations, it wouldn't be possible to find various potentials that approximate the square ones as close as one wants?

To answer these two questions it is necessary to go somewhat further into our discussion. Hence, we will postpone the solution of this apparent dilemma to the fourth section.

With respect to the solutions we get when we introduce the square potentials into the Schrödinger equation, we may say the following: when the value of k in expression (2) is real inside an infinite domain of the variable x, it is not possible to find a normalization for these functions—since they are not L^2. Indeed, not being L^2 these functions are not even members of the set of acceptable functions of the quantum formalism.

Under these circumstances, what we usually do is to appeal to a mathematical process called "box normalization" where we consider the particle as being in the interior of a cubic box of side L, and being normalized within this finite volume. We make the box volume tend to infinity keeping constant the overall probability density related with the functions. This is the mathematical way by means of which we reintroduce the functions (2) in the formal apparatus of quantum formalism.

Even if we decide to accept this mathematical process as licit, it is possible to ask if the use of such functions may be accommodated within the interpretation usually superimposed to the quantum formalism.

Indeed, being functions whose value of the energy is completely fixed (e.g. E), these functions, due to the Heisenberg dispersion relations, loose all its temporal localization, or saying differently, it is not possible to fix in any way, coherent with the principles of quantum mechanics, an initial time for the phenomenon.

Due to all we have said above, we have to absolutely refuse the objectivity of the phenomena described by the square potentials and their respective solutions.

It is, however, possible to argue that there exists other examples where the phenomenon of tunneling appears. One clear such example is the harmonic oscillator where, we may show, there exists a finite probability of finding particles in the classical forbidden regions (beyond the classical turning points), as explained by the usual interpretation.

In the following section we will present in a very brief way the formalism and interpretation usually associated with this problem.

3 Harmonic Oscillator

Within this application the potential is given by

$$V(x) = C^2 x^2/2, \tag{4}$$

where C is a constant. Clearly, this is an acceptable potential for the quantum formalism and our first complain of the last section is no longer valid.

The Schrödinger equation associated with this potential may be written as

$$\frac{d^2\psi(x)}{dx^2} + (\beta - \alpha^2)\psi(x) = 0, \tag{5}$$

where

$$\alpha \equiv 2\pi m\nu/\hbar \text{ and } \beta \equiv 2mE/\hbar, \tag{6}$$

with m the particle mass, E its energy, $\hbar$ Planck's constant and the frequency ν given by

$$\nu^2 = \frac{C^2}{4\pi^2 m}. \tag{7}$$

The normalized solutions for this problem are amply known and may be written as

$$\psi(x) = \left(\frac{\sqrt{\alpha}}{\sqrt{\pi}2^n n!}\right)^{1/2} e^{-\alpha x^2/2)} H_n(\sqrt{\alpha}x), \tag{8}$$

where $H_n(x)$ are the Hermite polynomials. The energies become

$$E_n = (n + 1/2)h\nu, \;\; n = 0, 1, 2, 3, ... \tag{9}$$

where $h = 2\pi\hbar$.

When looking at anyone of these solutions we may perceive that it exists a finite probability of finding particles in the regions beyond the classical turning points.

We also note that the functions are L^2 and hence are not subjected to our previous criticism, made in the last section.

It then seems to be an obvious conclusion the impossibility of presenting a classical interpretation to explain this phenomenon. Indeed, in the regions beyond the classical turning points, the kinetic energy has to be negative implying a complex momentum.

In the next section we will show how such an interpretation may be give.

4 Classical Tunnelling

For the same potential of the previous section we may write the hamiltonian function as

$$H = \frac{p^2}{2m} + \frac{1}{2}C^2x^2. \tag{10}$$

If H_0 is the initial energy of some particle when submitted to the potential (4), the classical turning points may be written as

$$x_{ret} = \pm\sqrt{2H_0}/C = \pm p_0/\sqrt{M}C, \tag{11}$$

where p_0 is its initial momentum. These points are obtained making the final momentum of the particle p_f=0.

Taking, without loss of generality, the fundamental state $n = 0$ in the quantum solutions obtained in the last section, we may write

$$H_0 = \hbar C/\sqrt{M} \text{ and } \psi_0(x) = \pi^{-1/4}\alpha^{1/4}e^{-\alpha x^2/2}. \tag{12}$$

The probability density in configuration space becomes

$$\rho(x) = (\alpha/\pi)^{1/2}e^{-\alpha x^2}. \tag{13}$$

Such probability density may be used to find the probability of having particles beyond the classical turning points. This probability may be written as

$$pr^Q = \int_{-\infty}^{-x_{ret}} \rho(x)dx + \int_{+x_{ret}}^{\infty} \rho(x)dx, \tag{14}$$

which gives, performing the integrals,

$$pr^Q = 1 - erf(\alpha^{1/2}x_{ret}), \tag{15}$$

where $erf(x)$ stays for the error function.

Using now the infinitesimal Wigner-Moyal transformation, as defined in our previous papers[1]-[12], we may get the normalized classical probability distribution function, defined upon phase space, as

$$F_0(x, p; t) = \frac{1}{\pi\hbar}e^{-ax^2-p^2/a\hbar^2}. \tag{16}$$

One has to note, however, that this classical distribution implies that there exists a dispersion in the momentum Δp —and also in the energy

(as we have already pointed out[9]). In this case, within the interpretation scheme proposed by this series of papers, we have to accept that there exists particles in the *ensemble* with various values of momenta[1].

We may then ask what is the probability of finding particles within this *ensemble* with initial momentum p_i greater then p_0. Such a calculation is performed by means of the integral

$$pr^{cl} = 2\int_{p_0}^{\infty} F_0(p)dp, \tag{17}$$

where

$$F_0(p) = \int_{-\infty}^{\infty} F_0(x,p;t)dx. \tag{18}$$

It is easy to show that the result of this integration gives

$$pr^{cl} = 1 - erf(\frac{p_0}{\sqrt{\alpha}\hbar}). \tag{19}$$

To have the two results coinciding we must have

$$\frac{p_0}{\sqrt{\alpha}\hbar} = \sqrt{\alpha}x_{ret}. \tag{20}$$

Noting that

$$\alpha\hbar = C\sqrt{m}, \tag{21}$$

we get

$$x_{ret} = \frac{p_0}{\sqrt{m}C}, \tag{22}$$

which is precisely the expression (classical) to the classical turning point obtained in (11).

This result signifies that, inside the original *ensemble*, whose mean kinetic energy whose energy was H_0, there were a number of particles whose momentum module was greater or equal to the momentum p_0, necessary to go beyond the classical turning points (derived from this *mean energy*). Hence,

[1]Note that it is not essential to go to the classical distribution. The quantum distribution in momentum space itself may be used. The calculations with this function will give, however, exactly the same results, since this distribution is equivalent to the lateral distribution in momentum space derived from the classical function, as was showed in the appendix A of our first paper[1].

nothing more obvious (and classical) than having these particles going beyond those classical turning points, since their initial energy is greater than the energy used to derive these points.

The phenomenon may be completely explained from the classical statistical point of view, without appealing to the negative kinetic energy concept. or any other pathology.

We may now answer the questions made in the second section with respect to the adequacy of using square potentials to simulate real potentials. We have to consider carefully the fact that some potential, not being of the square type, no matter how close it is of this format, will never allow solutions whose energy is free from dispersion. In this case, it will always be possible to find in the *ensemble* a certain number of particles with high values of the initial momentum p_i, even in a very small number (i.e. with small probability). These are the particles that will surmount the potential barrier. The solutions of the Schrödinger equation will be absolutely acceptable.

The choice of square potentials simplifies too much the problem denying us to identify all the subtleties related to it and inducing us to interpret its formal results in a mistaken manner. Their use in the scope of quantum theory is then subjected to great criticism. One way to overcome partially the problem is to use superpositions of plane waves which allow us to simulate an initial distribution function that presents the desired deviations.

5 Conclusions

In this brief paper we have shown how one may explain, inside the formalism of quantum mechanics and within the classical statistical *ensemble* interpretation, the widely known phenomenon of tunneling.

As a consequence of this explanation, it was demonstrated that this effect is just a statistical effect related to the fact of having, as a main characteristic of the quantum formalism, probability densities with non-zero root mean square deviations, implying that it will always be possible to find, within such an *ensemble,* particles with high values of the momentum. Even if this is highly improbable in a particular case, it is the probability of finding such particles which will determine the effect of "tunneling".

As was shown this effect may hardly continue to be called tunnel effect since there is nothing being "tunneled".

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